BOUNDARY VALUE MAPS, SZEGÖ MAPS AND INTERTWINING OPERATORS

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ABSTRACT. We consider one series of unitarizable representations, the cohomological induced modules $A_{\mathfrak{q}}(\lambda)$ with dominant regular infinitesimal character. The minimal K-type (τ, V) of $A_{\mathfrak{q}}(\lambda)$ determines a homogeneous vector bundle $V_\tau \longrightarrow G/K$. The derived functor modules can be realized on the solution space of a first order differential operator $\mathcal{D}_{\mathfrak{l}}^\lambda$ on V_τ. Barchini, Knapp and Zierau gave an explicit integral map $\mathcal{S}$ from the derived functor module, realized in the Langlands classification, into the space of smooth sections of the vector bundle $V_\tau \longrightarrow G/K$. In this paper we study the asymptotic behavior of elements in the image of $\mathcal{S}$. We obtain a factorization of the standard intertwining opeartors into the composition of the Szegö map $\mathcal{S}$ and a passage to boundary values.

0. INTRODUCTION

Let G be a linear, connected, semisimple Lie group and let K be a maximal compact subgroup of G. We assume that rank G = rank K. In this situation the cohomologically induced representation $A_{\mathfrak{q}}(\lambda)$ with dominant regular infinitesimal character can be realized on solution spaces of invariant differential equations in G/K. See ([Wo], [B-2]).

The minimal K-type (τ, V) of $A_{\mathfrak{q}}(\lambda)$ determines an homogeneous vector bundle $V_\tau \longrightarrow G/K$. Corresponding to certain choices of positive roots, there is defined a certain homogeneous, first order differential operator $\mathcal{D}_{\mathfrak{l}}^\lambda$ on V_τ. In [BKZ] and [B-1] the authors gave an explicit nonzero integral map $\mathcal{S}$ from the derived functor module, realized in the Langlands classification, to the space of smooth sections of the vector bundle $V_\tau \longrightarrow G/K$. The image of $\mathcal{S}$ is annihilated by the differential operator $\mathcal{D}_{\mathfrak{l}}^\lambda$.

When the representation under consideration is in the discrete series of G, the differential operator $\mathcal{D}_{\mathfrak{l}}^\lambda$ coincides with Schmid's differential operator [S]. In this case the $\mathcal{S}$-transform was introduced by Knapp and Wallach in [KW].

The realization of irreducible unitary representations in the G/K-picture generalizes Bargmann's formulation of holomorphic discrete series of $SU(1,1)$, [Bar]. In terms of semisimple Lie groups, functions on the circle suggest non-unitary principal series and holomorphic functions on the disk suggest discrete series representations. The $\mathcal{S}$-transform in [KW] reduces in limiting cases (limits of discrete series for $SU(1,1)$) to the Szegö kernel for the disk. See [K-1]. Thus, we call $\mathcal{S}$ a Szegö map.

In this paper we study the asymptotic behavior of functions in the image of the Szegö map and use that information to define a "boundary value map" from the image of $\mathcal{S}$ into an appropriate principal series representation of G. We obtain a factorization of the standard intertwining operator into the composition of the Szegö map, as in [BKZ] and [B-1], and a passage to boundary values. Sections 6 and 7 contain the main theorems and definitions of the paper. We use known results on the asymptotic behavior of Eisenstein integrals, and the theory on the constant term in [HC-1]. The interpretation of these results for specific cases that correspond to Szegö maps is new.

The Szegö maps, as in [KW], [Bl-1], [BKZ], [B-1], can be regarded as vector valued versions of the Poisson integrals studied by, among others, Kashiwara et al. in [KKMOOT]. We observe that in [KKMOOT] the composition of the boundary value map and the Poisson transform is a standard intertwining operator.

Brian Blank in [Bl-1], [Bl-2], [Bl-3] has studied Szegö maps and boundary value behavior of limits of discrete series. For groups of real rank one he showed the boundary value map as a factor of the projection operator given by the Knapp-Stein intertwining operator. Blank's treatment of boundary value maps for groups with real rank greater than one uses deep results of Harish-Chandra.

To describe the main results more precisely, we introduce some notation. Let G be a linear connected semisimple Lie group with complexification $G^{\mathbb{C}}$, let T be a torus in K, and L be the centralizer of T in G. We denote the Lie algebras of the Lie groups G, K, T, L by $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{t}_0$, $\mathfrak{l}_0$, etc. and their complexification by $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}, \mathfrak{l}$, etc. We write the Cartan decomposition of $\mathfrak{g}_0$ relative to the Cartan involution θ as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Our θ-stable parabolic subalgebra of $\mathfrak{g}$ is $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ with $\mathfrak{u}$ the unipotent radical.

Extend $\mathfrak{t}_0$ to a maximal abelian subspace $\mathfrak{b}_0$ of $\mathfrak{k}_0$. Let λ be in $\mathfrak{b}^*$.

We build, as in [BKZ] and [B-1], a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{l}_0 \cap \mathfrak{p}_0$ and denote by Σ_L and Σ_G the sets of restricted roots for L and G. Let Σ_L^+ and Σ_G^+ be the sets of positive restricted roots (in an ordering that will be specified later). We let ρ_L and ρ_G stand for half the sum of the roots in Σ_L^+ and Σ_G^+. In [BKZ] and [B-1] it was shown that Σ_G^+ can be defined so that ρ_L is Σ_G^+-dominant.

Define $\mathfrak{n}_0$ to be the sum of the root spaces in $\mathfrak{g}_0$ for the positive restricted roots, $\bar{\mathfrak{n}}_0$ to be $\theta\mathfrak{n}_0$, and N and $\bar{N}$ to be the corresponding analytic subgroups. Put $A = \exp \mathfrak{a}_0$ and let $P = MAN$ be the canonically associated cuspidal parabolic subgroup of G.

Under a positivity condition on λ, we associate to $A_{\mathfrak{q}}(\lambda)$ a non unitary principal series representation, $\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$, where σ is a representation in the discrete series of M. The Szegö map is an integral operator that carries $\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$ continuously in G-equivariant fashion into $C^\infty(G/K, V_\tau)$.

Let $A^+ = \{a \in A | \alpha(\log a) > 0, \text{ for every } \alpha \in \Sigma_G^+\}$. We define the boundary value $\mathcal{B}$ of K-finite elements $[\mathcal{S}f]$ in the image of the Szegö map by restricting them to $K/(M \cap K) \times A^+$ and computing the limit

$$(0.1) \qquad \lim_{(a_+ \to \infty, P)} a_+^{\rho_G - \rho_L} \mathcal{S}f(ka_+^{-1}).$$

When ρ_L is regular and dominant, then the limit (0.1) can be easily computed by using obsevation (6.4) and Remark 7.2 in this paper combined with standard techniques such as the Dominated Convergence Theorem. The limit provides a "pointwise boundary value" embedding of the Zuckerman modules into the reducible non

unitary principal series representations, $\text{ind}_{MAN}^{G}(\sigma, -\rho_L, 1)$. When real rank $G =$ real rank L, the Boundary Value map is a G-equivariant map.

The case of singular ρ_L is more delicate. Inspired by the study of the asymptotic expansion near the walls of matrix coefficients of admissible representations, as presented in [K-3], p. 247, we build a parabolic subgroup $P_1 = M_1 A_1 N_1$ such that $P \subset P_1$, $P_1 \cap M = M A_{M_1} N_{M_1}$ is a cuspidal parabolic subgroup of M_1 and such that $\rho_L|_{\mathfrak{a}_1}$ is regular and dominant. The idea is first to redo a certain amount of the regular case, replacing objects associated to P by objects associated to P_1 in order to compute

$$(0.2) \qquad \lim_{a_1 \to \infty, P_1} a_{M_1}^{\rho_{M_1}} a_1^{\rho_1 - \rho_L} \mathcal{S}f(k a_{M_1} a_1)$$

for a fixed $a_{M_1} \in A_{M_1}$. (When the real ranks are different, we must consider the limit (0.2) in the L^2-sense. See §7.) The second step is to observe that very well known results on the asymptotic behavior of Eisenstein integrals give information about the asymptotic expansion of (0.2) relative to $P_1 \cap M$. We define the boundary value of $\mathcal{S}f$ as the leading terms in the asymptotic expansion of (0.2) relative to $P_1 \cap M$.

Many results concerning the topics touched upon here exist in the literature. Szegö kernels associated to discrete series originated in [K-1] and [KW]. Szegö kernels associated to limits of discrete series were studied in [Bl-1] and [Bl-2]. Szegö kernels associated to $A_{\mathfrak{q}}(\lambda)$ appear in [BKZ], [B-1] and [B-2]. Casselman has obtained embedding results by means of asymptotic coefficients [C], [M]. Bargmann established the embedding of Limit of discrete series of $SU(1,1)$. Knapp-Okamoto [K-O] has constructed for limits of holomorphic discrete series an embedding that is the exact analogue of the Hardy embedding. Among other contributions to the topic are [GKST], [Ka], [Ma], [Mi], [Ta].

1. Preliminaries

Our underlying group G is assumed to be linear, connected, semisimple, with rank $G =$ rank K. The equal rank assumption will allow us to refer to the results in [B-2] with no further considerations. We continue with K, θ, L, T, and various Lie algebras as in the introduction.

Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{b})$ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{b}$. We say that a root is compact or noncompact according as its root vector lies in $\mathfrak{k}$ or $\mathfrak{p}$, and we write Δ_c and Δ_n for the set of compact and noncompact roots in Δ.

Since $\mathfrak{b}_0$ centralizes $\mathfrak{t}_0$, $\mathfrak{b}_0$ is contained in $\mathfrak{l}_0$. Thus, we can speak of sets of roots $\Delta(\mathfrak{u}, \mathfrak{b})$, $\Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{b})$, $\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{b})$ as well as similar sets for $\mathfrak{l}$ and $\overline{\mathfrak{u}}$.

Our choice of a positive system $\Delta^+(\mathfrak{g}, \mathfrak{b})$ is so that $\Delta^+(\mathfrak{g}, \mathfrak{b}) = \Delta^+(\mathfrak{l}, \mathfrak{b}) \cup \Delta(\mathfrak{u}, \mathfrak{b})$, with $\Delta^+(\mathfrak{l}, \mathfrak{b})$ as in [B-1]. The roots in $\overline{\mathfrak{u}}$ are the negative of the roots in $\mathfrak{u}$. For any of these sets of roots, we let $\delta()$ be half the sum of their members. In particular

$$(1.1) \qquad\qquad \delta(\mathfrak{u} \cap \mathfrak{k}) + \delta(\mathfrak{u} \cap \mathfrak{p}) = \delta(\mathfrak{u}).$$

We choose root vectors E_α for the roots in $\Delta(\mathfrak{g}, \mathfrak{b})$, so that the Killing form satisfies

$$(1.2) \qquad\qquad C(E_\alpha, E_{-\alpha}) = 1.$$

We define H_α to be the member in $\mathfrak{b}$ dual to α under the Killing form. Then, by (1.2) it follows that $[E_\alpha, E_{-\alpha}] = H_\alpha$.

It is important to observe that in [BKZ] and [B-1] the positive system $\Delta^+(\mathfrak{l}, \mathfrak{b})$, was given along with *a compatible sequence* $\alpha_1, \alpha_2 \ldots \alpha_l$ of roots in $\Delta_n(\mathfrak{l}, \mathfrak{b})$. Thus, our choice of Δ^+ carries with it a choice of a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$ and a sequence of strongly orthogonal roots $\alpha_1, \ldots, \alpha_l$ used in defining $\mathfrak{a}_0$. In terms of these roots we can define, as in [BKZ], a Cayley transform $\mathbf{c}$ and construct a new θ-stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathbf{c}(\mathfrak{b})$. The subalgebra $\mathfrak{h}_0$ is in $\mathfrak{l}_0$ since $\mathbf{c}$ is in $\mathrm{Ad}L^{\mathbf{C}}$. (See [BKZ].) Under the definition $(\mathbf{c}\beta)(H) = \beta(\mathbf{c}^{-1}H)$; $\mathbf{c}$ carries $\Delta(\mathfrak{g}, \mathfrak{b})$ to $\mathbf{c}\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. For $\beta \in \Delta(\mathfrak{g}, \mathfrak{b})$ we write $X_\beta = \mathbf{c}(E_\beta)$. The reader should be warned that X_β is a root vector for the root $\mathbf{c}\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$.

Let

$$(1.3) \qquad\qquad \begin{aligned} \mathfrak{b}_0' &= \sum_{j=1}^{l} \mathbf{R}(iH_{\alpha_j}) \subset \mathfrak{b}_0, \\ \mathfrak{b}_0^- &= \text{ orthocomplement of } \mathfrak{b}_0'' \text{ in } \mathfrak{b}_0. \end{aligned}$$

We denote by $\mathfrak{m}_0$ the orthocomplement of $\mathfrak{a}_0$ in the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$. The subalgebra $\mathfrak{b}_0^-$ is a compact Cartan subalgebra of $\mathfrak{m}_0$. If $\Delta^- = \{\beta \in \Delta(\mathfrak{g}, \mathfrak{b}) : \beta|_{\mathfrak{b}'} = 0\}$, then the root system decomposition of $\mathfrak{m}$ is

$$(1.4) \qquad\qquad \mathfrak{m} = \mathfrak{b}^- \oplus \sum_{\beta \in \Delta_-} \mathbf{C}X_\beta.$$

Each vector X_β with $\beta \in \Delta_-$ is either in $\mathfrak{k}$ or $\mathfrak{p}$, and we call $\mathbf{c}\beta$ compact or non-compact accordingly. We define $\Delta^+(\mathfrak{m}, \mathfrak{b}) = \{\mathbf{c}\beta : \beta \in \Delta_- \cap \Delta^+(\mathfrak{g}, \mathfrak{b})\}$.

We choose a positive system $(\mathbf{c}\Delta)^+$ for $\mathbf{c}\Delta = \Delta(\mathfrak{g}, \mathfrak{b})$ different from the image of Δ^+ under $\mathbf{c}$. Namely, we list $H_{\alpha_1}, \ldots H_{\alpha_l}$ as an ordered orthogonal basis of $i\mathfrak{b}_0'$. We use this basis in lexicographic fashion to determine positivity for members of $\mathbf{c}\Delta$ that do not vanish on $\mathfrak{a}$. (Namely, $\mathbf{c}\beta \in (\mathbf{c}\Delta)^+$ if $\beta(H_{\alpha_1}) > 0$ or $\beta(H_{\alpha_1}) = 0$ and $\beta(H_{\alpha_2}) > 0$, etc.) For roots supported in $\mathfrak{b}^-$ we say that $\mathbf{c}\beta \in (\mathbf{c}\Delta)^+$ if $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{b})$. We can consistently define positive restricted roots by saying that $\mathbf{c}\beta|_{\mathfrak{a}} > 0$ if $\mathbf{c}\beta|_{\mathfrak{a}} \neq 0$ and $\mathbf{c}\beta \in (\mathbf{c}\Delta)^+$.

We keep the notation in the introduction for various sets of restricted roots and positive restricted roots.

It is crucial to observe that the sequence of roots $\alpha_1, \ldots, \alpha_l$ can be chosen so that ρ_L is dominant for G. (See [BKZ, §12] and [B-1].)

We can associate a parabolic subgroup to our Cartan subalgebra $\mathfrak{h}$. Let $\mathfrak{n}_0$, $\overline{\mathfrak{n}}_0$, N, $\overline{N}$ and A be as in the introduction. Let M^0 be the analytic subgroup with Lie algebra $\mathfrak{m}_0$. Write $M = Z_K(\mathfrak{a})M^0$. Observe that $\mathfrak{b}^-$ is a maximal abelian subspace

of $\mathfrak{k} \cap \mathfrak{m}_0$ and $B^- = \exp \mathfrak{b}^-$ is a maximal torus in $K \cap M^0$. The parabolic subgroup $P = MAN$ is cuspidal.

Let $M^\# = M^0 Z_M$ be the product of the identity component and the center of M. Lemma 12.30 in [K-3] shows that

$$(1.4) \qquad\qquad M^\# = M^0 F(B^-),$$

where $F(B^-)$ is the finite abelian group in the center of M. In particular $F(B^-) \cap M^0$ is central in M^0. Consequently we have

$$(1.5) \qquad\qquad F(B^-) \cap M^0 = F(B^-) \cap B^-.$$

It is important to note that the parabolic $P = MAN$ has A contained in L. Thus, $L \cap MAN = (L \cap M)A(L \cap N)$ is a minimal parabolic subgroup of L. Furthermore, it is not difficult to see that ([B-1, (4.1)])

$$F(B^-) \subset L \cap K.$$

Remark 1.1. If $\alpha_1, \dots, \alpha_l$ denotes the strongly orthogonal sequence of non-compact roots used in defining $\mathfrak{a}_0$, then $\mathbf{c}\alpha_i \in \Delta(\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{b}^-)$ and $\mathbf{c}\alpha_i|_{\mathfrak{b}^-} = 0$. Moreover, $\frac{2\langle \mathbf{c}\alpha_i, \mathbf{c}\alpha_i \rangle}{|\alpha_i|^2} = 2 \neq +1$. Thus, $\mathbf{c}\alpha_i|_{\mathfrak{a}}$ are *useful odd roots* in the sense of [K-3, Chap. XIV]. By [K-3, Th. 14:39], $\omega_0 = s_{\mathbf{c}\alpha_1} \dots s_{\mathbf{c}\alpha_l}$ is an element of the Weyl group $W(G : A)$. Moreover, by [K-3, Th. 14:46] ω_0 admits a representative p in K. For such a p, $\mathrm{Ad}(p)$ acts as the identity on $\mathfrak{m}_0$. Hence, $\omega_0 \in W(G : A)$ is so that $\omega_0(\rho_L) = -\rho_L$ and ω_0 "acts" as the identity on $\mathfrak{m}_0$.

2. Principal series parameters

When the dominance condition

$$\langle \lambda, \alpha \rangle \geqq 0 \text{ for all } \alpha \in \Delta^+$$

on λ is satisfied, we associate to the Zuckerman module $A_\mathfrak{q}(\lambda)$ a principal series representation, according to the Langlands classification. The construction is done in detail in [B-1].

Let $P = MAN$ be the Langlands decomposition of the cuspidal parabolic subgroup introduced in §1. Put $\Psi = \Delta^+(\mathfrak{g}, \mathfrak{h}) \cap \Delta^-$, and let $\mathbb{D}(\Psi, \lambda)$ denote the discrete series of M^0 with Blattner parameter $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})|_{\mathfrak{b}^-}$ and infinitesimal character $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}) + \delta_K(\Psi) - \delta_n(\Psi)$.

Next we define a representation in the discrete series of $M^\# = M^0 F(B^-)$ that agrees with $\mathbb{D}(\Psi, \lambda)$ on M^0. To do so it is enough to define a character χ on $F(B^-)$ compatible with the action of $\mathbb{D}(\Psi, \lambda)$ on $F(B^-) \cap B^-$. Thus, $\mathbb{D}(\Psi, \lambda) \otimes \chi$ is a well defined representation of $M^\#$.

Since $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is integral, $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is the differential of a well defined character $\xi_{\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})}$ of B. We define χ on $F(B^-)$ by means of

$$(2.1) \qquad\qquad \chi(z) = \xi_{\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})}(uzu^{-1})$$

where $\mathbf{c} = \mathrm{Ad}u$ is the Cayley transform in §1.

We take the representation σ, in the discrete series of M, to be

$$(2.2) \qquad\qquad \sigma = \mathrm{ind}_{M^\#}^M(\mathbb{D}(\Psi, \lambda) \otimes \chi).$$

We call χ the central character of σ.

We use the $\mathfrak{a}'$-parameter ρ_L and the discrete series representation σ to form the principal series representation $\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$. This is the representation in which G acts by the left regular representation on the space

$$(2.3) \qquad \{f : G \longrightarrow V^\sigma | f(xman) = a^{-\rho_L - \rho_G}\sigma(m)^{-1}f(x)\}$$

where V^σ is the space where σ acts. ($U(P, \sigma, \rho_L)(g)f(x) = f(g^{-1}x)$.)

We apply double induction formulas to show that our principal series representation is of the form

$$(2.4) \qquad \mathrm{ind}_{M^\# AN}^G([\mathbb{D}(\Psi, \lambda) \otimes \chi], \rho_L, 1).$$

Let (V_χ, τ_χ) be an irreducible K-representation with highest weight $\chi = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ and highest weight vector ϕ.

Lemma 2.1. *The cyclic span of ϕ in V_χ under $K \cap M^\#$, $\tilde{V}_\chi$, is irreducible under $K \cap M^\#$. Namely,*

(a) *ϕ is a highest weight vector under $\mathfrak{m} \cap \mathfrak{k}$, with highest weight $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})|_{\mathfrak{b}^-}$ and,*

(b) *$\mathbf{C}\phi$ is a one dimensional space stable under the action of $L \cap K$ (which contains $F(B^-)$).*

Proof. See [BKZ], Prop. 3.1. $\square$

The discrete series representation $\mathbb{D}(\Psi, \lambda) \otimes \chi$ can be obtained as solution spaces of differential operators acting on $C^\infty(M^\#/K \cap M^\#, \tilde{V}_\chi)$. Indeed, following [KW] we can identify the discrete series representation $\mathbb{D}(\psi, \lambda) \otimes \chi$ with the representation of $M^\#$ acting on a subspace $\mathcal{J}_{M^\#}^\chi$ of the kernel of Schmid's differential operator $\mathcal{D}_{M^\#}$.

Lemma 2.2. *If ω_0 is as in Remark 1.1, then the representations σ and $\omega_0\sigma$ are equivalent.*

Proof. Let p denote the representative of ω_0 in K. The discrete series $\omega_0\sigma$ is defined as $\omega_0\sigma(m) = \sigma(p^{-1}mp)$. The central character of $\omega_0\sigma$ is $\omega_0\chi$. (Note that $\omega_0\chi$ is well defined since the action of $W(G : A)$ on the center of M is well defined.) In Remark 1.1, we observed that $\mathrm{Ad}\, p$ acts as the identity on $\mathfrak{m}_0$. Thus, $\omega_0\mathbb{D}(\psi, \lambda)$ is equivalent to $\mathbb{D}(\psi, \lambda)$ as an M^0-representation. In view of Proposition 4.9 in [K-3], we need to show that $\omega_0\chi = \chi$ in order to prove the lemma.

By Lemma 2.1 (b), χ is the parameter of the one dimensional representation of $F(B^-)$ acting on $\mathbf{C}\phi$. The group $F(B^-)$ is spanned by the elements $\gamma_\beta = \exp\frac{2i\pi h_\beta}{|\beta|^2}$ where β runs through the restricted roots and where h_β is the member of $\mathfrak{a}$ dual to β. For $\gamma_\beta \in F(B^-)$ we have

$$(2.5) \qquad \tau(\gamma_\beta)\phi = c\phi.$$

Thus, it is enough to identify the scalar c in equation (2.5). Let $\{\alpha_1, \ldots, \alpha_l\}$ be the sequence of strongly orthogonal roots introduced in §1. Write $h_{\alpha_j} = \mathrm{Ad}u_{\alpha_j}^{-1}H_{\alpha_j}$. We can expand

$$h_\beta = \sum \frac{\langle \beta, \mathbf{c}\alpha_i \rangle}{|\mathbf{c}\alpha_i|^2} h_{\alpha_i},$$

then can copy the argument in ([KW], page 344) to show that c is given by (-1) raised to the power $\sum \frac{2\langle \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}), \alpha_j \rangle}{|\alpha_j|^2} \cdot \frac{\langle \beta, \mathbf{c}\alpha_j \rangle}{|\beta|^2}$, with the sum extended over those j

for which $\frac{2\langle\beta,\mathbf{c}\alpha_j\rangle}{|\mathbf{c}\alpha_j|^2}$ is even. Thus,

$$(2.6) \quad \chi(\gamma_\beta) = \xi_{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})}(\gamma_\beta) = \exp\left(i\pi\sum_j \frac{\langle\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p}),\alpha_j\rangle}{|\alpha_j|^2}\cdot\frac{2\langle\beta,\mathbf{c}\alpha_j\rangle}{|\alpha_j|^2}\right).$$

On the other hand,

$$(2.7) \quad \begin{aligned}(s_{\mathbf{c}\alpha_k}\chi)(\gamma_\beta) =&\exp\left(\sum_{j\neq k}i\pi\frac{\langle\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p}),\alpha_j\rangle}{|\alpha_j|^2}\cdot\frac{2\langle\beta,\mathbf{c}\alpha_j\rangle}{|\alpha_j|^2}\right)\\ &\times\exp\left(-i\pi\frac{\langle\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p}),\alpha_k\rangle}{|\alpha_k|^2}\cdot\frac{2\langle\beta,\mathbf{c}\alpha_k\rangle}{|\alpha_k|^2}\right).\end{aligned}$$

The sum on the right hand side of (2.7) runs over the j so that $\frac{2\langle\beta,\mathbf{c}\alpha_j\rangle}{|\alpha_j|^2}$ is even. By (2.6) and (2.7), we have

$$(s_{\mathbf{c}\alpha_k}\chi)(\gamma_\beta) = \chi(\gamma_\beta).$$

$\square$

3. Knapp-Stein intertwining operators

To facilitate exposition and for the convenience of the reader we include in this section some standard results on intertwining operators . References for this section are [K-2], [KS], [KZ], [K-3] and [W-1].

Let $\mathrm{ind}_{MAN}^G(\pi,\nu,1)$ be a principal series representation induced from a tempered M-representation π. Let $w \in W(G:A)$ be so that $w\nu = -\overline{\nu}$, $[w\pi] = [\pi]$ and $w^2 = I$.

Since $w\pi$ is equivalent to π, it is possible to define $\pi(w)$ as an operator on V_π in exactly two ways [K-2, page 546]. $\pi(w)$ is a multiple of an operator exhibiting $w\pi$ and π as unitarily equivalent.

Lemma 3.1. *Let $\nu \in \mathfrak{a}_0'$ be such that $\mathrm{Re}\langle\nu,\alpha\rangle > 0$ for all $\alpha \in \Sigma_G^+$, and let π be a tempered representation of M. If f is a K-finite function in $\mathrm{ind}_{MAN}^G(\pi,\nu,1)$ and ψ is an $M\cap K$-finite vector in V_π, then*

$$\pi(w)\int_{\bar{N}}\langle f(xp\bar{n}),\psi\rangle d\bar{n} < \infty$$

for all $x \in G$.

Proof. See [K-2, §7] or [W-1, 5.3.1].

$\square$

Remark 3.2. When P is a minimal parabolic subgroup of G and ν is so that $\mathrm{Re}\langle\nu,\alpha\rangle > 0$, for all $\alpha \in \Sigma^+$, the integral

$$\int_{\bar{N}}f(xp\bar{n})d\bar{n}$$

converges uniformly for every continuous function in $\mathrm{ind}_{MAN}^G(\pi,\nu,1)$.

We retain the above assumptions on π and ν.

Lemma 3.3. *If f is a K-finite vector in $\mathrm{ind}_{MAN}^G(\pi,\nu,1)$, then there exists a finite dimensional subset $V(f)$ in $V_{M\cap K-finite}^\pi$ such that*

$$\int_{\bar{N}}\langle f(xp\bar{n}),\psi\rangle_{V_\pi}d\bar{n} = 0$$

for all $\psi \in V_{M\cap K-finite}^\pi$ orthogonal to $V(f)$.

Proof. See [W-1, 5.3.2].   □

Lemma 3.4. *There exists a $(\mathfrak{g}, K)$-intertwining operator $\pi(w)A(w, \pi, \nu)$ from $\mathrm{ind}_{MAN}^{G}(\pi, \nu, 1)$ to $\mathrm{ind}_{MAN}^{G}(\pi, -\bar{\nu}, 1)$ so that for each K-finite function f, $k \in K$ and $\psi \in V_{M \cap K-finite}^{\pi}$*

$$\langle \pi(w)A(w, \pi, \nu)f(k), \psi \rangle_{V^{\pi}} = \int_{\bar{N}} \langle \pi(w)f(kp\bar{n}), \psi \rangle_{V_{\pi}} d\bar{n}.$$

Proof. See [K-2, §7], [W-1, 5.3.3].   □

Let π be an irreducible tempered representation of M. If $P_1 = MAN_1$ and $P_2 = MAN_2$ are two parabolic subgroups of G, then we consider

(3.3) $$A(P_2, P_1, \pi, \nu)f = \int_{\bar{N}_1 \cap N_2} f(k\bar{n})d\bar{n}$$

It is possible to show ([W-2, §10], page 13) existence of a constant c_{π} such that whenever $\langle \mathrm{Re}\nu, \alpha \rangle \geqq c_{\pi}$ for every α with $\mathfrak{g}_{\alpha} \in \mathfrak{n}_1 \cap \bar{\mathfrak{n}}_2$, then the integral (3.3) converges uniformly.

Lemma 3.5. *There exists a function $\eta(P, \bar{P}, \pi, \nu)$, meromorphic in $\nu \in \mathfrak{a}'$, such that*

$$A(P, \bar{P}, \pi, \nu)A(\bar{P}, P, \pi, \nu) = \eta(P, \bar{P}, \pi, \nu)I$$

whenever the left hand side is defined.

Proof. See [K-2, §XIV] and [W-2, 10.5.4].   □

Remark 3.6. For a definition of the function $\eta(P, \bar{P}, \pi, \nu)$ see [K-2, §XIV, Proposition 14.13] or [W-2, 10.5.7]. If $s \in W(G : A)$ and $k \in K$ is a representative of s, then $\eta(P, \bar{P}, k\pi, s\nu) = \eta(P, \bar{P}, \pi, \nu)$. (See [W-2, page 45].)

Following Knapp in [K-2, §14], we define complex valued functions $\gamma(P, \bar{P}, \pi, \nu)$ meromorphic on $\mathfrak{a}'$ so that

(3.4) $$\gamma(P, \bar{P}, \pi, \nu)\overline{\gamma(\bar{P}, P, \pi, \nu)} = \eta(P, \bar{P}, \pi, \nu).$$

Normalized intertwining operators can be defined by means of

(3.5)
$$\begin{aligned} \mathcal{A}(P, \bar{P}, \pi, \nu) &= \gamma(P, \bar{P}, \pi, \nu)^{-1}A(P, \bar{P}, \pi, \nu), \\ \mathcal{A}_P(w, \pi, \nu) &= \gamma(w^{-1}Pw, P, \pi, \nu)^{-1}A_P(w, \pi, \nu). \end{aligned}$$

Lemma 3.7. *The operators $\mathcal{A}(P, \bar{P}, \pi, \nu)$ and $\mathcal{A}_P(w, \pi, \nu)$ extend meromorphically as functions of ν. They have no pole at ν when $\mathrm{Re}\,\nu = 0$.*

Proof. See [K-2], Prop. 14.20 and Prop. 14.21.   □

4. Szegö kernels

The Szegö map $\mathcal{S}$ is an integral operator that carries the representation $\mathrm{ind}_{M^{\#}AN}^{G}([\mathbb{D}(\Psi, \lambda) \otimes \xi], \rho_L, 1)$ continuously in G-equivariant fashion into the space of smooth V_{τ}-valued sections over G/K. Under the identification mentioned after Lemma 2.1 in §2, the principal series representation is the representation on which

G acts by the left translation on the space

$$\mathbf{C}^\infty(G, [\mathbb{D}(\Psi, \lambda) \otimes \xi], \rho_L, 1)$$
$$= \{f \in \mathbf{C}^\infty(G, \mathcal{J}_{M\#}^\chi)| \text{ with transformation law under } M^0 F(B^-)AN\}$$

with $\mathcal{J}_{M\#}^\chi \subset \mathrm{Ker}(\mathcal{D}_{M\#})$. Elements of the latter space give rise to elements of $\mathrm{ind}_{K \cap M\# AN}^G(\tilde{\tau}_\chi, \rho_L, 1)$ by means of the map $g \to g^\#$, where $g^\#(x) = g(x)(1)$.

Theorem 4.1. *The operator $\mathcal{S}$ given by*

$$(\mathcal{S}f)(x) = \int_K \tau_\chi(k) f^\#(xk) dk$$

carries $\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$ *continuously in G-equivariant fashion into* $\mathbf{C}^\infty(G/K, V_\chi)$.
The representation of $(\mathfrak{g}, K)$ acting on the space of K-finite vectors in the image of $\mathcal{S}$ is infinitesimally equivalent to the Zuckerman module $A_\mathfrak{q}(\lambda)$.

Proof. See [BKZ], [B-1], [B-2], [Wo].   □

Remark 4.2. In [Wo], Wong showed (under a positivity condition on λ) that certain Dolbeault cohomology spaces $H^s(G/L, \mathbf{C}_\lambda^\#)$ are admissible Fréchet G-modules of finite length. Moreover, these cohomology modules are the maximal globalization of the $A_\mathfrak{q}(\lambda)$.

In [BKZ] and [B-1], the authors constructed an explicit integral operator $\mathcal{P}$ (a kind of Penrose transform) carrying the space of Dolbeault cohomology in G-equivariant fashion into the space of smooth V_τ-valued smooth sections on G/K. Thus, Image $\mathcal{P}$ admits a natural Fréchet topology and in this topology can be regarded as the maximal globalization of $A_\mathfrak{q}(\lambda)$.

In [B-2], the author identifies a special subspace in the image of $\mathcal{P}$. This is the subspace of sections that are in the image of the Szegö map $\mathcal{S}$. The action of $(\mathfrak{g}, K)$ on the space of K-finite vectors in the image of $\mathcal{S}$ is infinitesimally equivalent to $A_\mathfrak{q}(\lambda)$. In particular, the image of $\mathcal{S}$ is an admissible module of finite length.

The $(\mathfrak{g}, K)$ module $A_\mathfrak{q}(\lambda)$ is infinitesimally equivalent to the Langlands quotient of the principal series representation $\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$ introduced in §2. The Langlands classification is usually phrased a little differently, however. Define, as in [VZ],

$$Z = \{\alpha \text{ a root of } \mathfrak{a} \in \mathfrak{g}|\langle \rho_L, \alpha \rangle = 0\},$$
$$A_1 = \bigcap_{\alpha \in Z} \ker \alpha \subset A,$$
$$M_1 A_1 = \text{ Langlands decomposition of the centralizer of } A \text{ in } G \ ,$$
$$\nu = \rho_L|_{\mathfrak{a}_1}.$$

We notice that $P \cap M_1 = MA_{M_1} N_{M_1}$ is a cuspidal parabolic subgroup of M_1. (When real rank G = real rank L, $P \cap M_1$ is a minimal parabolic subgroup of M_1. In general, $P \cap M_1$ contains a minimal parabolic subgroup of M_1.) Define

$$\pi_{\sigma, \nu} = \mathrm{ind}_{P \cap M_1}^{M_1}(\sigma, 0, 1).$$

Since $\langle \rho_L, \alpha \rangle \geq 0$ for all roots α of $\mathfrak{a}$ in $\mathfrak{n}$, there is a unique parabolic subgroup of G having Levi factor $M_1 A_1$ and containing P. We write

(4.3) $P \subset P_1 = M_1 A_1 N_1.$

By induction in stages,

$$(4.4) \qquad \operatorname{ind}_{MAN}^G(\sigma, \rho_L, 1) = \operatorname{ind}_{M_1 A_1 N_1}^G(\operatorname{ind}_{P \cap M_1}^{M_1}(\sigma, 0, 1), \rho_L|_{\mathfrak{a}_1}, 1).$$

The isomorphism (4.4) is implemented by the map

$$(4.5) \qquad f \longrightarrow F, \quad F(x)(m) = f(xm).$$

By the choice of P, $\langle \rho_L, \alpha \rangle > 0$ for all roots α of $\mathfrak{a}_1$ in $\mathfrak{n}_1$. Since π is unitarily induced from a discrete series, it is tempered. It can be shown that π is irreducible. Therefore, $\operatorname{ind}_{M_1 A_1 N_1}^G(\pi, \rho_L|_{\mathfrak{a}_1}, 1)$ is the kind of representation considered in [BW], and it has a unique Langlands quotient, $A_{\mathfrak{q}}(\lambda)$.

We identify the principal series representations in (4.4) with

$$\operatorname{ind}_{M^\# AN}^G(\mathbb{D}(\psi, \lambda) \otimes \xi, \rho_L, 1) = \operatorname{ind}_{M_1^\# A_1 N_1}^G(\operatorname{ind}_{M_1^\# \cap P}^{M_1^\#}(\mathbb{D}(\psi, \lambda) \otimes \xi, 0, 1), \rho_L|_{\mathfrak{a}_1}, 1).$$

Here $M_1^\# = M_1^0 F_1$, where F_1 is the finite abelian group generated by elements $\gamma_\beta = \exp(i\pi |\beta|^{-2} H_\beta)$ with β a root in $\Delta(\mathfrak{g}, \mathfrak{a}_1 \oplus \mathfrak{a}_{M_1} \oplus \mathfrak{b}^-)$ vanishing on $\mathfrak{a}_1 \oplus \mathfrak{b}^-$. It follows readily from its definition that $F_1 \subset F(B^-) \subset L \cap K$. (Compare with [BKZ, §3].)

Lemma 4.3. *Let (V_χ, τ_χ) be an irreducible K-representation with highest weight $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ and highest weight vector ϕ. The cyclic span of ϕ in V_χ under $K \cap M_1^\#$, $(\tilde{V}_\chi^1, \tau_\chi^1)$, is irreducible under $K \cap M_1^\#$.*

Proof. Since $F_1 \subset L \cap K$, we can copy the proof of Lemma 3.1. $\qquad \square$

Since the representation $\pi_{\sigma, 0}$ is irreducible, we have

Lemma 4.4. *The Szegö map*

$$\tilde{\mathcal{S}} : \operatorname{ind}_{P \cap M_1^\#}^{M_1^\#}(\sigma, 0, 1) \longrightarrow C^\infty(M_1^\#/(K \cap M_1^\#), \tilde{V}_\chi^1)$$

given by

$$(4.6) \qquad \tilde{\mathcal{S}}g(m) = \int_{K \cap M_1^\#} \tau_\chi^1(k)[g(mk)(1)]\, dk$$

is an $M_1^\#$-equivariant isomorphism onto its image.

We observe that $K \cap M^\#$ admits an invariant Haar measure, and the integral (4.6) is well defined. We denote by $\mathcal{J}_{M_1^\#}^\chi$ the image of $\tilde{\mathcal{S}}$. Define a G-equivariant operator $\mathcal{S}_1 : \operatorname{ind}_{M_1 A_1 N_1}^G(\mathcal{J}_{M_1^\#}^\chi, \rho_L|_{\mathfrak{a}_1}, 1) \to C^\infty(G/K, V_\chi)$ by means of

$$(4.7) \qquad \mathcal{S}_1 H(x) = \int_K \tau_\chi(k)[H(xk)(1)]\, dk \quad \text{for } x \in G.$$

Remark 4.5. If $f \in \operatorname{ind}_{M^\# AN}^G(\mathbb{D}(\psi, \lambda) \otimes \xi, \rho_L, 1)$, then the function H given by

$$H(x)(m) = \int_{K \cap M_1^\#} \tau_\chi^1(l)[f(xml)(1)]\, dl$$

$$= \int_{K \cap M_1^\#} \tau_\chi^1(l)[F(x)(ml)(1)]\, dl$$

for $x \in G$ and $m \in M_1^\#$, lives in $\text{ind}_{M^\# A_1 N_1}^G(\mathcal{J}_{M_1^\#}^\chi, \rho_L|_{\mathfrak{a}_1}, 1)$, and

$$\begin{aligned}
\mathcal{S}_1 H(x) &= \int_K \tau(k) \int_{K \cap M_1^\#} \tau_\chi^1(l)[f(xkl)(1)] dl dk \\
&= \int_K \int_{K \cap M_1^\#} \tau(kl)[f(xkl)(1)] dl dk \\
&= \int_K \tau(k)[f(xk)(1)] dk \\
&= \mathcal{S}f(x)
\end{aligned}$$

5. Irreducible tempered representations

In this section we recall several results of Harish-Chandra concerning the harmonic analysis of real reductive groups . The principal references for this section are [HC-1], [HC-2] and §12 in [W-2]. We have attempted to keep our notation in accordance with that of [W-2] insofar as that was possible. We include a special version of those theorems related to the study of the asymptotic behavior of $K \cap M_1$-finite matrix coefficients of the irreducible tempered representation $\pi_{\sigma, i\nu} = \text{ind}_{M_1 \cap P}^{M_1}(\sigma, i\nu, 1)$ with $\nu \in \mathfrak{a}_0'$.

To each $K \cap M_1$-finite vector w in $\text{ind}_{M_1 \cap P}^{M_1}(\sigma, i\nu, 1)$ we associate a $K \cap M_1$-finite continuous linear functional on the space of smooth vectors in the principal series by means of

$$(5.1) \qquad \lambda_w(\pi_{\sigma, i\nu}(g)v) = \int_{K \cap M_1} \langle \pi_{\sigma, i\nu}(g)v(m), w(m) \rangle_{V_\sigma} dm.$$

Here v stands for a smooth vector in the space of realization of the induced representation. Since the representation $\pi_{\sigma, i\nu}$ is tempered, then we have

$$(5.2) \qquad \|\lambda_w(\pi_{\sigma, i\nu}(g)v)\| \leqq C_{v, w} \Xi_{M_1}(g),$$

where Ξ_{M_1} denotes the Harish-Chandra spherical function for M_1.

Let $A_{M_1}^+ = \{a \in A_{M_1} | \alpha(\log a) > 0 \text{ for all } \Sigma_{M_1}^+\text{-dominant roots } \alpha\}$, and let $\mathcal{A}_w(M_1)$ denote the span of smooth functions on M_1 satisfying the conditions in [W-2, page 147].

If $a \in A_{M_1}^+$ and $m \in M_1$, then we denote by $\Phi_{P \cap M_1}(a, \lambda_w, \pi_{\sigma, i\nu}(m)v)$ the leading term in the asymptotic expansion of $\lambda_w(\pi_{\sigma, i\nu}(am)v)$. (See [W-1, §4] and [W-2, §12, page 145].)

Proposition 5.1.

(1) $\Phi_{P \cap M_1} \in \mathcal{A}_w(M_1)$.
(2) $\lim_{a_{M_1} \to \infty, P \cap M_1} \|a_{M_1}^{\rho_{M_1}} \lambda_w(\pi_{\sigma, i\nu}(a_{M_1} m)v) - \Phi_{P \cap M_1}(a_{M_1}, \lambda_w, \pi_{\sigma, i\nu}(m)v)\| = 0$.
(3) $\Phi_{P \cap M_1}(., ., .)$ *is the unique function in $\mathcal{A}_w(M_1)$ satisfying (2).*
(4) $\Phi_{P \cap M_1}(a_{M_1}, \lambda, \pi_{\sigma, i\nu}(m)v)$ *is continuous on ν and v and real analytic in (a, ν).*

Proof. See [HC-1, §21] and [W-2, 12.3.5]. ($\lim_{a_{M_1} \to \infty, P \cap M_1}$ means that $\log a$ tends to infinity in $\mathfrak{a}_{M_1}^+$ in such a way that every positive root, applied to $\log a$, tends to infinity.) $\qquad\square$

The leading term $\Phi_{P \cap M_1}(a, \lambda, \pi_{\sigma, i\nu}(m)v)$ can be related to standard intertwining operators. Indeed, if $W(M_1 : A_1) = N_{K \cap M_1}(A_{M_1})/Z_{K \cap M_1}(A_{M_1})$, then

$$(5.3) \qquad \phi_{P \cap M_1}(a, \lambda, \pi_{\sigma, i\nu}(m)v) = \sum_{s \in W(M_1, A_{M_1})} c_s(m) a^{is\nu}$$

provided $\nu \in \mathfrak{a}'_0$ is regular.

We set

$$(5.4) \qquad \Phi_{P \cap M_1, s}(\lambda, \pi_{\sigma, i\nu}(m)v) = c_s(m).$$

See [W-2], page 159.

Proposition 5.2. *If $\nu \in \mathfrak{a}'_0$ is regular, then $A(\bar{P} \cap M_1, P \cap M_1, \sigma, .)$ is holomorphic at $i\nu$ and*

$$\Phi_{P \cap M_1, 1}(\lambda_w, \pi_{\sigma, i\nu}(m)v) = \gamma_{P \cap M_1}^{-1} \langle A(\bar{P} \cap M_1, P \cap M_1, \sigma, i\nu)\pi_{\sigma, i\nu}(m)v(1), w(1)\rangle_{V_\sigma}$$

with $\gamma_{P \cap M_1} = \int_{\bar{N}_1} a(\bar{n}_1)^{-2\rho_{M_1}} d\bar{n}_1$.

Proof. See [W-2, 12.5.3] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let $s \in W(M_1, A_1)$, and let $k \in K \cap M_1$ be a representative of s.

Proposition 5.3. *With the notation just introduced, if $\nu \in \mathfrak{a}'_0$ is regular, then*

$$\Phi_{P \cap M_1, s}(\lambda_w, \pi_{\sigma, i\nu}(m)v) = \gamma_{P \cap M_1}^{-1} \eta^{-1}(P \cap M_1, kPk^{-1} \cap M_1, k\sigma, is\nu)$$
$$\times \langle A(\bar{P} \cap M_1, P \cap M_1, k\sigma, is\nu) A(P \cap M_1, kPk^{-1} \cap M_1, k\sigma, is\nu)v(k^{-1}),$$
$$\times A(kP \cap M_1 k^{-1}, P \cap M_1, k\sigma, is\nu)w(k^{-1}\rangle_{V_\sigma}.$$

Proof. See [W-2, page 163]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 5.4. The functions $\gamma^{-1}(\bar{P} \cap M_1, P \cap M_1, \sigma, i\nu)\Phi_{P \cap M_1, s}(\lambda_w, \pi_{\sigma, i\nu}(m)v)$ are holomorphic in $\nu \in \mathfrak{a}'_0$. Moreover,

$$\lim_{a_{M_1} \to \infty, P \cap M_1} a_{M_1}^{\rho_{M_1}} \gamma^{-1}(\bar{P} \cap M_1, P \cap M_1, \sigma, i\nu)\lambda_w(\pi_{\sigma, 0}(ma_{M_1})v)$$

exists and is given by

$$\sum_{s \in W(M_1, A_{M_1})} \phi_{P \cap M_1, s}(\lambda_w, \pi_{\sigma, 0}(m)v)\gamma^{-1}(\bar{P} \cap M_1, P \cap M_1, \sigma, 0).$$

6. Boundary value maps: the case real rank G = real rank L

In this section we study the asymptotic behavior of elements in the image of the Szegö map and use this information to define a boundary value map. Throughout this section we shall assume that real rank G = real rank L. This assumption implies that the parabolic subgroup $P = MAN$ introduced in §2 is minimal. In particular M is compact.

Let $A^+ = \{a \in A | \alpha(\log a) > 0$ for all $\sum_G^+$-dominant $\alpha\}$. As a motivation we observe that A^+ is an open subset of the set of regular elements A' of G. $KA'K$ is an open dense submanifold of G and the map $(kM, a) \to kaK$ defines an analytic isomorphism of $K/M \times A^+$ onto $KA'K/K$ which is an open dense submanifold of G/K.

Let $\beta_1, \ldots, \beta_l$ be the simple roots of $\sum_G^+$ and let $h_{\beta_1}, \ldots, h_{\beta_l}$ be the dual basis in $\mathfrak{a}$. For any $\mathfrak{a} \in A^+$, we define $y_i(a) = e^{-\beta_i(\log a)}, 1 \leqq i \leqq l$. The map

$$A^+ \longrightarrow (0,1)^l,$$

(6.1)
$$a \longrightarrow y(a) = (y_1(a), \ldots, y_l(a))$$

is an analytic isomorphism. Let $\Omega = K/M \times (-1,1)^l$ and let $\Omega_+ = K/M \times (0,1)^l$. Then, $K/M \times A^+$ is analytically isomorphic to Ω_+. We think of $K/M \times A^+$ as a cone or "wedge" in Ω with "edge" $K/M \times \{0\}$. We will refer to K/M as the "Martin boundary" of G/K. In view of this observation it is natural to define the boundary value of $\mathcal{S}f$ in the image of $\mathcal{S}$ by means of

(6.2)
$$\lim_{a_+ \to \infty, P} a_+^{\rho_G - \rho_L} \mathcal{S}f(ka_+)$$

provided the limit exists. Here $\lim_{a_+ \to \infty, P}$ means $\log(a)$ tends to infinity in $\mathfrak{a}^+$ in such a way that every positive root, applied to $\log(a)$, tends to infinity.

When ρ_L is regular dominant, the domain of the Szegö map consists of smooth vectors in the induced representation $\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$.

If $w_0 = \pi s_{\mathbf{c}\alpha_i} \in W(G:A)$ and $p \in K$ is a representative, then the intertwining map

$$T(w_0) : V_\sigma \longrightarrow V_\sigma,$$

$$v \longrightarrow P_\sigma[\tau(w_0)^{-1}v],$$

where P_σ denotes the projection from V_τ to V_σ, exhibits the representations σ and $\omega_0\sigma$ of M as unitarily equivalent. We define

(6.3)
$$\sigma(w_0) = T(w_0).$$

Elements in the image of $\mathcal{S}$ are smooth sections of the vector bundle $V_\tau \to G/K$. Thus, $P_\sigma[\mathcal{S}f(ka_+p)] = P_\sigma[\tau(p^{-1})\mathcal{S}f(ka_+)] = \sigma(w_0)\mathcal{S}f(ka_+)$. On the other hand, $w_0^2 = 1$ and w_0 acts by $-I$ on $\mathfrak{a}$. Hence,

(6.4)
$$P_\sigma[\mathcal{S}f(ka_+)] = \sigma(w_0)\mathcal{S}f(kpa_+^{-1}).$$

It is formula (6.4) together with standard results that allow us to prove

Lemma 6.1. *If ρ_L is in the open restricted positive Weyl chamber, then*

(6.5)
$$\lim_{a_+ \to \infty, P} a_+^{\rho_G - \rho_L} P_\sigma[\mathcal{S}f(ka_+)] = \sigma(w_0)A(w_0, \rho_L, \sigma)f(k).$$

The limit (6.5) converges pointwise and uniformly for $k \in K$.

When ρ_L is singular, the analysis in Lemma 6.1 runs into difficulties. To avoid such difficulties we build a parabolic subgroup $P_1 = M_1 A_1 N_1$ with $P \subset P_1$ as in §4. If $a_{M_1} \in A_{M_1}^+$ and $a_1 \in A_1^+$, then $a_{M_1}a_1 \in A^+$. The idea is first to fix $a_{M_1} \in A_{M_1}^+$ and to redo a certain amount of Lemma 6.1, replacing objects associated to MAN by objects associated to $P_1 = M_1 A_1 N_1$. Indeed, by Remark 4.5, to $f \in \mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$ we can associate a function $H \in \mathrm{ind}_{M_1^\# A_1 N_1}^G(\mathcal{J}_{M_1}^X, \rho|_{\mathfrak{a}_1}, 1)$ so that

$$\mathcal{S}f(x) = \mathcal{S}_1 H(x) = \int_K \tau(k)[H(xk)(1)]dk$$

and

$$P_\sigma[\mathcal{S}f(ka_+)] = \sigma(w_0)\mathcal{S}f(ka_{M_1}pa_1^{-1})$$

$$= \sigma(w_0)\int_K \tau(l)[H(ka_{M_1}pa_1^{-1}l)(1)]dl.$$

We refer to Remark 4.5 for the definition of the function H.

We have three decompositions, $G = KM_1A_1N_1$, $G = KAN$ and $M_1 = (M_1\cap K)A_{M_1}N_{M_1}$. We let H_1, H, H_{M_1} be the respective H-functions and $\rho_1, \rho_G, \rho_{M_1}$ be the respective ρ-functions. Some relations among these are

$$\rho_G = \rho_1 + \rho_{M_1},$$
$$H = H_1 + H_{M_1},$$
$$\rho_G - \rho_L(H_1(x)) = \rho_1 - \rho_L(H_1(x)),$$
$$\rho_G - \rho_L(H_{M_1}) = \rho_1 - \rho_L(H_1(x)) + \rho_{M_1}(H_{M_1}(x)), \quad \text{since } \rho_1 - \rho_L(H_{M_1}(x)) = 0.$$

For each $x \in G$, we write $x = k(x)\mu_1(x)a_1(x)n_1(x)$ with $(k, \mu_1, a_1, n_1) \in K \times \exp(\mathfrak{m}_1 \cap \mathfrak{p}) \times A_1 \times N_1$.

An argument similar to the one in the proof of Lemma 6.1 gives us

Lemma 6.2. *With the notation just introduced,*

$$\lim_{a_1\to\infty,P_1} a_1^{\rho_1-\rho_L}\mathcal{S}_1H(ka_{M_1}pa_1^{-1}) = [A(\bar{P}_1,P_1,\pi_{\sigma,0},\rho_L|_{\mathfrak{a}_1})H](ka_{M_1}p)(1).$$

The limit converges pointwise for $k \in K$ and $a_{M_1} \in A_{M_1}^+$.

In order to define the boundary value of an element in the image of the Szegö map, in the ρ_L singular case, we shall introduce deformations $\mathcal{S}_\epsilon$ and $\mathcal{S}_{1,\epsilon}$ of our Szegö maps. Following the work of others, for example the work of Kashiwara et al. in [KKMOOT], we shall define the bounday value of $\mathcal{S}f$ to be the leading terms in the asymptotic expansion of

$$\lim_{a_1\to\infty,P_1} a_{M_1}^{\rho_{M_1}} a_1^{\rho_1-\rho_L}\sigma(\omega_0)\mathcal{S}_\epsilon f_\epsilon(ka_{M_1}pa_1)$$

relative to $P \cap M_1$, as $\epsilon \to 0$.

The results summarized in §5 will give us the desired leading term.

Proof of Lemma 6.1. We have

$$\mathcal{S}f(kpa_+^{-1}) = \int_K \tau_{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})}(l)f(kpa_+^{-1}l)dl.$$

We make the change of variables that passes from K to $\bar{N} \times M$, and observe that $f(kpa_+^{-1}\bar{n}) = e^{-\rho_L-\rho_G H(\bar{n})}f(kpa_+^{-1}\kappa(\bar{n}))$, to obtain

$$\mathcal{S}f(ka_+^{-1}) = a_+^{\rho_G+\rho_L}\int_{\bar{N}} e^{-\rho_G+\rho_L H(\bar{n})}\tau(\kappa(\bar{n}))f(kpa_+^{-1}\bar{n}a_+)d\bar{n}$$

$$= a_+^{\rho_G+\rho_L}\det(\mathrm{Ad}(a_+^{-1})|_{\bar{n}})^{-1}\int_{\bar{N}} e^{-\rho_G+\rho_L H(a_+\bar{n}a_+^{-1})}e^{-\rho_G-\rho_L H(\bar{n})}$$

$$\times \tau(\kappa(a_+\bar{n}a_+^{-1}))f(kp\kappa(\bar{n}))d\bar{n}.$$

On the other hand, $\det(\mathrm{Ad}(a_+^{-1}|_{\bar{n}})^{-1} = a_+^{-2\rho_G}$. Thus,

(6.6)

$$\mathcal{S}f(kpa_+^{-1}) = a_+^{-\rho_G+\rho_L}\int_{\bar{N}} e^{-\rho_G+\rho_L H(\bar{n}^{a+})}e^{-\rho_L-\rho_G H(\bar{n})}\tau(\kappa(\bar{n}^{a+}))f(kp\kappa(\bar{n}))d\bar{n}$$

where $\bar{n}^{a_+} = a_+ \bar{n} a_+^{-1}$ and $d\bar{n}$ is normalized so that $\int_{\bar{N}} e^{-2\rho_G H(\bar{n})} d\bar{n} = 1$. By (6.6), we have

(6.7)

$$\lim_{a_+ \to \infty, P} a_+^{\rho_G - \rho_L} P_\sigma [\mathcal{S} f(ka_+)]$$

$$= \lim_{a_+ \to \infty, P} \sigma(w_0) \int_{\bar{N}} e^{-\rho_G + \rho_L H(\bar{n}^{a_+})} e^{-\rho_G - \rho_L H(\bar{n})} \tau(\kappa(\bar{n}^{a_+})) f(kp\kappa(\bar{n})) d\bar{n}.$$

We use the dominated convergence theorem to show that the limit and integral in (6.7) can be interchanged. In fact, since f is a continuous function on K the integrand is (6.7) is dominated above by

(6.8)

$$\|e^{-\rho_G - \rho_L H(\bar{n})} e^{-\rho_G + \rho_L H(\bar{n}^{a_+})} \tau(\kappa(\bar{n}^{a_+})) f(kp\kappa(\bar{n}))\|$$

$$\leqq C e^{-\rho_L - \rho_G H(\bar{n})} e^{-\rho_G + \rho_L H(\bar{n}^{a_+})}.$$

On the other hand, $\bar{n}^{a_+} \in \bar{N}$, and the real parameter $\rho_G - \rho_L$ is Σ_G^+-dominant. Thus, the right hand side of (6.8) is bounded above by $e^{(-\rho_L - \rho_G)(H(\bar{n}))}$.

Since we have assumed that ρ_L is in the open positive restricted Weyl chamber and since our the parabolic subgroup $P = MAN$ is minimal, the function $e^{-\rho_G - \rho_L H(\bar{n})}$ is integrable. (See for example [K-3, Lemma 7.23].)   □

It is immediate to check that for each fixed $x \in G$, the pointwise limit

$$\lim_{(a_+ \to \infty, P)} a_+^{\rho_G - \rho_L} \mathcal{S} f(xa_+)$$

exists, and it is given by $\sigma(w_0) A(w_0, \rho_L, \sigma) f(x)$. Thus, we can reinterpret the boundary value map as a G-equivariant embedding of the image of $\mathcal{S}$ into the principal series representation $\mathrm{ind}_{MAN}^G(\sigma, -\rho_L, 1)$. In other words, the diagram of G-equivariant maps

$$\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1) \xrightarrow{\sigma(w_0) A(w_0, \sigma, \rho_L)} \mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1)$$
$$\searrow \qquad \swarrow$$
$$\mathrm{Im}(\mathcal{S}) \subset C^\infty(G/K, V_\chi)$$

commutes. Since Image $\mathcal{S}$ is irreducible, then $\mathcal{B}$ is an injective map.

Proof of Lemma 6.2. We make the change of variables that passes from K to $\bar{N}_1 \times (M_1 \cap K)$, and observe that

$$H(ka_{M_1} pa_1^{-1} \bar{n}_1)(1) = e^{-\rho_L - \rho_1 (H_1(\bar{n}_1))} H(ka_{M_1} pa_1^{-1} \kappa(\bar{n}_1))(\mu_1(\bar{n}_1)),$$

in order to show that

$$\mathcal{S}_1 H(ka_{M_1} pa_1^{-1})$$

$$= a_1^{\rho_1 + \rho_L} \int_{\bar{N}_1} e^{-\rho_1 + \rho_L H_1(\bar{n}_1)} \tau(\kappa(\bar{n}_1)) \{ H(ka_{M_1} pa_1^{-1} \bar{n}_1 a_1)(\mu_1(\bar{n}_1)^{-1}) \} d\bar{n}_1.$$

By a change of variables we have

(6.9)

$$\lim_{a_1 \to \infty, P_1} a_1^{-\rho_1 - \rho_L} \mathcal{S}_1 H(ka_{M_1} pa_1^{-1})$$

$$= \lim_{a_1 \to \infty, P_1} \int_{\bar{N}_1} e^{-\rho_1 + \rho_L H_1(\bar{n}_1^{a_1})} e^{-\rho_1 - \rho_L H_1(\bar{n}_1)}$$

$$\times \tau(\kappa(\bar{n}_1^{a_1})) \{ H(ka_{M_1} p\kappa(\bar{n}_1))(\mu_1(\bar{n}_1) \mu_1(\bar{n}_1^{a_1})^{-1}) \} d\bar{n}_1$$

where $\overline{n}_1{}^{a_1} = a_1 \overline{n}_1 a_1^{-1}$. The limit (6.9) is computed by means of standard techniques. For example see Wallach's exposition on a result of Langlands in [W-1, 5.3.4]. For the benefit of the non-experts, we briefly indicate how to proceed.

By Remark 4.5 in §4,

$$H(ka_{M_1} p\kappa(\bar{n}_1))(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1})$$
$$= \int_{K\cap M_1^{\#}} \tau_\chi^1(m)\{F(ka_{M_1} p\kappa(\bar{n}_1))(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1}m)\}dm_1$$

where F is a function in $\operatorname{ind}_{M_1^{\#} A_1 N_1}^G (\operatorname{ind}_{P\cap M_1^{\#}}^{M_1^{\#}}(\sigma,0,1), \rho_L|_{a_1}, 1)$. Moreover, if $\{\phi_i\}$ is an orthonormal basis for V_σ, then

$$
\begin{aligned}
&H(ka_{M_1} p\kappa(\bar{n}_1))(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1})\\
(6.10)\quad &= \sum_i \int_{K\cap M_1^{\#}} \langle F(ka_{M_1} p\kappa(\bar{n}_1)(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1}m), \tau_\chi^1(m^{-1})\phi_i\rangle dm_1 \quad \phi_i.
\end{aligned}
$$

Define $\psi_i(m) = \tau_\chi^1(m^{-1})\phi_i$ for $m \in M_1^{\#}\cap K$. The functions ψ_i give rise to $M_1^{\#}\cap K$-finite vectors in $\pi_{\sigma,0}$. Thus, we can rewrite (6.10) as

$$
\begin{aligned}
&H(ka_{M_1} p\kappa(\bar{n}_1))(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1})\\
&= \sum_i \overline{\langle \pi_{\sigma,0}(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1})\psi_i, F(ka_{M_1} p\kappa(\bar{n}_1))(.)\rangle}\phi_i.
\end{aligned}
$$

The representation $\pi_{\sigma,0}$ is tempered, so by (5.1) we conclude that

$$(6.11)\qquad \|H(ka_{M_1} p\kappa(barn_1)(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1})\| \leqq C\Xi_{M_1}(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1}).$$

Thus, the integral on the right hand side of (6.9) is bounded above by

$$(6.12)\qquad \int_{\bar{N}_1} e^{-\rho_1+\rho_L H_1(\bar{n}_1^{a_1})}e^{-\rho_1+\rho_L H_1(\bar{n}_1)}\Xi_{M_1}(\mu_1(\bar{n}_1)\mu_1(\bar{n}_1^{a_1})^{-1})d\bar{n}_1.$$

Since $\rho_L|_{\mathfrak{a}_1}$ is in the positive restricted Weyl chamber, the integral (6.12) converges uniformly for $a_1 \in A_1^+$. See [HC-1, page 49], [L, Lemma 3.12]. Hence, we can interchange limit and integration in (6.9). $\qquad\square$

Corollary 6.3. *Let f be a vector in $\operatorname{ind}_{M^\# AN}^G(\sigma, \rho_L, 1)$ and let F be the corresponding function in $\operatorname{ind}_{M_1^\# A_1 N_1}^G (\operatorname{ind}_{M_1^\#\cap P}^{M_1^\#}(\sigma,0,1), \rho_L|_{\mathfrak{a}_1}, 1)$. If $P = MAN$ is a minimal parabolic subgroup of G, then for each $a_{M_1} \in A_{M_1}$, we have*

$$
\begin{aligned}
&\lim_{a_1\to\infty, P_1} a_{M_1}^{\rho_{M_1}} a_1^{\rho_1-\rho_L}\mathcal{S}f(ka_{M_1}a_1)\\
&= \sum_i \{\int_{\bar{N}_1} a_{M_1}^{\rho_{M_1}}\langle \pi_{\sigma,0}(a_{M_1})F(kp\bar{n}_1)(.), \psi_i(.)\rangle_{L^2(K\cap M_1^\#)}d\bar{n}_1\}\tau(p^{-1})\phi_i.
\end{aligned}
$$

With the notation of sections 5 and 6, we have

Proposition 6.4. *For each $\bar{n}_1 \in \bar{N}_1$, let $\Phi_{P\cap M_1^\#}(a_{M_1}, \psi_i, F(kp\bar{n}_1)(.))$ be the leading term in the analytic expansion of $\langle \pi_{\sigma,0}(a_{M_1})F(kp\bar{n}_1)(.), \psi_i(.)\rangle_{L^2(K\cap M_1^\#)}$. Then,*

when real rank G = real rank L, we have

(6.13)
$$\lim_{a_{M_1}\to\infty, P\cap M_1^{\#}} \| \lim_{a_1\to\infty, P_1} a_{M_1}^{\rho_{M_1}} a_1^{\rho_1-\rho_L} \mathcal{S}f(ka_{M_1}a_1)$$
$$- \sum_i \{\int_{\bar{N}_1} \Phi_{P\cap M_1^{\#}}(a_{M_1}, \psi_i, F(kp\bar{n}_1)(.))d\bar{n}_1\}\tau(p^{-1})\phi_i\| = 0.$$

Proof. The convergence of the integrals in (6.13) follows from well known results on the asymptotic behavior of the leading terms $\Phi_{P\cap M_1^{\#}}(.,.,.)$. The estimate in [W-2, Lemma 12.3.2 (1)], followed by an argument similar to the one in [W-1, 4.5.6] shows that such integrals are bounded above by

$$\int_{\bar{N}_1} e^{-\rho_1(1+\epsilon)H_1(\bar{n}_1)} \Xi_{M_1}(\mu_1(\bar{n}_1))d\bar{n}_1,$$

which converges by [HC-1, Corollary of Lemma 32.2]. By Lemma 6.2 and linearity of $\Phi_{P\cap M_1^{\#}}(a_{M_1}, \psi_i, v)$ on V, it follows that (6.13) is bounded above by

(6.14)
$$\lim_{a_{M_1}\to\infty, P\cap M_1^{\#}} \sum_i \int_{\bar{N}_1} e^{-\rho_L-\rho_1 H_1(\bar{n}_1)}$$
$$\times \|a_{M_1}^{\rho_{M_1}} \langle\pi_{\sigma,0}(a_{M_1}\mu_1(\bar{n}_1)^{-1}))F(kp\kappa(\bar{n}_1))(.), \psi_i(.)\rangle_{L^2(K\cap M_1^{\#})}$$
$$- \Phi_{P\cap M_1^{\#}}(a_{M_1}, \psi_i, \pi_{\sigma,0}(\mu_1(\bar{n}_1)^{-1})F(kp\kappa(\bar{n}_1))\|d\bar{n}_1.$$

By [W-2, Lemma 12.3.2 (3)], the integrand in (6.14) is dominated by

(6.15)
$$e^{-\rho_L-\rho_1 H_1(\bar{n}_1)}(1+\|\log\mu_1(\bar{n}_1)^{-1}\|)^d\Xi_{M_1}(\mu_1(\bar{n}_1)).$$

We proceed as in the first part of the proof to show that (6.15) belongs to $L^1(\bar{N}_1)$. Thus, we can interchange limit and integration. Now the proposition follows from Prop. 5.1 (2). $\square$

Let $\epsilon > 0$, and consider the principal series representation

$$\mathrm{ind}_{M^{\#}AN}^G(\sigma, \rho_L + \epsilon\rho_{M_1}, 1).$$

For $f_\epsilon \in \mathrm{ind}_{M^{\#}AN}^G(\sigma, \rho_L + \epsilon\rho_{M_1}, 1)$ let

$$F_\epsilon \in \mathrm{ind}_{M_1^{\#}A_1N_1}^G(\mathrm{ind}_{M_1^{\#}\cap P}^{M_1^{\#}}(\sigma, \epsilon\rho_{M_1}, 1), \rho_L|_{\mathfrak{a}_1}, 1) \text{ be } F_\epsilon(x)(m) = f_\epsilon(xm).$$

Define H_ϵ by

$$H_\epsilon(x)(m) = \int_{K\cap M_1^{\#}} \tau_\chi^1(m_1)F_\epsilon(x)(mm_1)dm_1$$
$$= \sum_i \langle\pi_{\sigma,i\epsilon\rho_{M_1}}(m^{-1})F_\epsilon(x)(.), \psi_i(.)\rangle_{L^2(K\cap M_1^{\#})}\phi_i.$$

Define

$$\mathcal{S}_\epsilon f_\epsilon(x) = \int_K \tau_\chi(k)f_\epsilon(xk)dk,$$
$$\mathcal{S}_{1,\epsilon}H_\epsilon(x)(m) = \int_K \tau_\chi(k)H_\epsilon(xk)(m)dk.$$

We can show as in Remark 4.5 that $\mathcal{S}_\epsilon f_\epsilon(x) = \mathcal{S}_{1,\epsilon}H_\epsilon(x)(1)$. $\mathcal{S}_\epsilon$ is a deformation of the Szegö map $\mathcal{S}$ in that $\mathcal{S}_0 = \mathcal{S}$.

Definition 6.5. For a fixed $x \in G$, define the normalized boundary value of $H_\epsilon(x)(.)$ to be

$$[\mathcal{B}_{M_1} H_\epsilon(x)](m)$$
$$= \sum_j \gamma^{-1}(\bar{P} \cap M_1^\#, P \cap M_1^\#, \sigma, i\epsilon\rho_{M_1}) \Phi_{P \cap M_1^\#, 1}(\psi_j, \pi_{\sigma, i\epsilon\rho_{M_1}}(m^{-1}) F_\epsilon(x)(.))$$

with $\Phi_{P \cap M_1^\#, 1}(., ., .)$ as in §5 (5.4).

By Proposition 5.2 we have

$$\mathcal{B}_{M_1}[H_\epsilon(x)](1)$$
$$= \gamma^{-1}_{P \cap M_1^\#} \sum_i \langle \mathcal{A}(\bar{P} \cap M_1^\#, P \cap M_1^\#, \sigma, i\epsilon\rho_{M_1}) F_\epsilon(x)(1), \phi_i \rangle_{V^\sigma} \phi_i$$
$$= \gamma^{-1}_{P \cap M_1^\#} \mathcal{A}(\bar{P} \cap M_1^\#, P \cap M_1^\#, \sigma, i\epsilon\rho_{M_1}) F_\epsilon(x)(1).$$

Definition 6.6. Define the boundary value of $\mathcal{S}_\epsilon f_\epsilon$ by means of

$$\mathcal{B}[\mathcal{S}_\epsilon f_\epsilon](k) = \mathcal{B}_{M_1}[\lim_{a_1 \to \infty, P_1} a_1^{\rho_1 - \rho_L} \sigma(w_0) \mathcal{S}_\epsilon f_\epsilon(k a_{M_1} p a_1)].$$

By Proposition 5.2 and [W-2, page 191] we have

$$\mathcal{B}[\mathcal{S}_\epsilon f_\epsilon](k) = \gamma^{-1}_{P \cap M_1^\#} \sigma(w_0) A_{P_1}(w_0, \pi_{\sigma, i\epsilon\rho_{M_1}}, \rho_L|_{\mathfrak{a}_1})$$
$$\circ \mathcal{A}(\bar{P} \cap M_1^\#, P \cap M_1^\#, \sigma, i\epsilon\rho_{M_1}) F_\epsilon(k)(1)$$
$$= \sigma(w_0) \mathcal{A}_P(w_0, \sigma, \rho_L + i\epsilon\rho_{M_1}) f_\epsilon(k).$$

Definition 6.7. For $f \in \text{ind}_{M^\# AN}^G(\sigma, \rho_L, 1)$, we define the normalized boundary value of $\mathcal{S}f$ to be

$$\mathcal{B}[\mathcal{S}f](k) = \lim_{\epsilon \to 0} \mathcal{B}[\mathcal{S}_\epsilon f_\epsilon](k)$$

Thus, $\mathcal{B}[\mathcal{S}f](k) = \sigma(w_0) \mathcal{A}_P(w_0, \sigma, \rho_L) f(k).$

Remark 6.8. The limit

$$\lim_{\epsilon \to 0} \lim_{a_{M_1} \to \infty, P \cap M_1^\#} a_{M_1}^{\rho_{M_1}}$$
$$\times \{\gamma^{-1}(\bar{P} \cap M_1^\#, P \cap M_1^\#, \sigma, i\epsilon\rho_{M_1}) \lim_{a_1 \to \infty, P_1} a_1^{\rho_1 - \rho_L} \sigma(w_0) \mathcal{S}_\epsilon f_\epsilon(k a_{M_1} p a_1^{-1})\}$$

exists and can be given explicitly in terms of intertwining operators by using Proposition 5.3 in §5.

7. BOUNDARY VALUES OF SZEGÖ INTEGRALS: THE DIFFERENT REAL RANK CASE

In this section we drop the assumption real rank G = real rank L. In the spirit of §6, we consider

$$(7.1) \qquad \lim_{a \to \infty, P} a_+^{\rho_L - \rho_G} \mathcal{S}f(k a_+).$$

Since the cuspidal parabolic subgroup P is not a minimal parabolic subgroup of G, in order to make sense of the limit (7.1) we restrict our attention to K-finite elements in Image $\mathcal{S}$.

The domain of the Szegö map consists of smooth vectors in

$$\mathrm{ind}_{M^\# AN}^G(\mathbb{D}(\psi,\lambda) \otimes \xi_{\lambda+2\delta(\mathfrak{u}\cap p)}, \rho_L, 1)$$

where $\mathbb{D}(\psi,\lambda) \otimes \xi_{\lambda+2\delta(\mathfrak{u}\cap p)}$ is the $M^\#$-discrete series representation constructed in §2. As indicated in §2, we identify the representation $\mathbb{D}(\psi,\lambda) \otimes \xi$ with the representation of $M^\#$ acting on a subspace $\mathcal{J}_{M^\#}^\chi$ of the kernel of Schmid's differential operator $\mathcal{D}_{M^\#}$.

Let $w_0 = \pi s_{\mathbf{c}\alpha_i} \in W(G, A)$ and let $p \in K$ be a representative. The intertwining map

$$[T(w_0)f](m) = P_\chi[\tau(p^{-1})f(m)],$$

where P_χ denotes the projection from V_χ to $\tilde{V}_\chi$, exhibits the representations of $M^\#$ acting on $C^\infty(M^\#/(K \cap M^\#), \tilde{V}_\chi)$ and $C^\infty(M^\#/p(K \cap M^\#)p^{-1}, \tilde{V}_{p(K\cap M^\#)p^{-1}})$ as equivalent.

Remark 7.1. Let f be a vector in $C^\infty(G, [(\mathbb{D}(\psi,\lambda) \otimes \xi], \rho_L, 1)$. For each element $k \in K, f(k)(.)$ is a smooth section of the vector bundle

$$M^\# \times_{M^\#\cap K} \tilde{V}_\chi \to M^\#/(M^\# \cap K).$$

Moreover, $f(k)(m)$ as a function of m lies in a subspace $\mathcal{J}_{M^\#}^\chi$ of the kernel of the Schmid's operator $\mathcal{D}_{M^\#}$.

If we assume that f is a K-finite vector in $C^\infty(G, [(\mathbb{D}(\psi,\lambda) \otimes \chi], \rho_L, 1)$, then $f(k)(.) \subset \{M^\# \cap K$-finite vectors in $\mathcal{J}_{M^\#}^\chi\}$. (See, [W-1, 5.2].) It follows from work of Hotta-Parthasarathy [HP], Schmid [S], Knapp-Wallach [KW], that $M^\#\cap K$-finite vectors in $\mathcal{J}_{M^\#}^\chi$ are square integrable.

Let $l \in K$ and let $\psi_1, \dots, \psi_d$ be a basis for span $\{R(M^\# \cap K)L(M^\# \cap K)f\}$. (Here, R and L denote right and left translation respectively.) Now, if $m_1, m_2 \in M^\# \cap K$ and $a^* \in A^*$, then

$$f(l)(m_1 a^* m_2) = \sum h_j(l)(m_1, m_2)\psi_j(a^*)$$

with $h_j(l) \in C^\infty((M^\# \cap K) \times (M^\# \cap K), \tilde{V}_\chi)$. Thus, there exists a constant C such that

$$|f(l)(m_1 a^* m_2)| \leqq C \sum |\psi_j(a_+^*)|$$

for all $l \in K, m_1, m_2 \in (M^\# \cap K)$ and $a^* \in A_+^*$. Our previous observations imply that the smooth $(M^\#\cap K)$-finite functions ψ_j are so that $X\psi_j$ are square integrable for all $X \in \mathcal{U}(\mathfrak{m})$. The asymptotic behavior of such functions is well understood. See for example [W-1, 5A 3.4]. We use the results in [W-1, 5A 3.4] and well known properties of the zonal spherical function Ξ_M of M to conclude that $f(l)(m)$ is bounded above by $\Xi_M(m)$.

Remark 7.2. Let $w_0 = \pi s_{\mathbf{c}\alpha_i} \in W(G, A)$ and let $p \in K$ be a representative. Then

$$\mathcal{S}f(kma_+) = \tau(p)\mathcal{S}f(kpm_1a_+^{-1}), \quad \text{with } m_1 = p^{-1}mp.$$

The observations in Remarks 7.1 and 7.2 combined with an argument similar to the one used in Lemma 6.2 allow us to prove

Proposition 7.3. *If ρ_L is in the open positive restricted Weyl chamber and f is a K-finite vector of $C^\infty(G, [\mathbb{D}(\psi, \lambda) \otimes \chi], \rho_L, 1)$, then*

(8.3)
$$\lim_{a_+ \to \infty, P} a^{\rho_G - \rho_L} P_\chi \mathcal{S} f(kma_+) = [T(w_0) A_P(w_0, \rho_L, \mathbb{D}(\psi, \lambda) \otimes \xi) f(k)](p^{-1}mp)$$

for each $m \in M^\#$. The limit (6.9) converges pointwise and uniformly for $k \in K$.

As in §6 we can reinterpret the boundary value map as a (now infinitesimal) pointwise embedding of Image $\mathcal{S}$ in the principal series $\mathrm{ind}_{MAN}^G(\sigma, -\rho_L, 1)$ (provided ρ_L is strictly dominant). Once again, the diagram of $(\mathfrak{g}, K)$-equivariant maps

$$\mathrm{ind}_{MAN}^G(\sigma, \rho_L, 1) \xrightarrow{T(w_0) A_P(w_0, \sigma, \rho_L)} \mathrm{ind}_{MAN}^G(w_0\sigma, -\rho_L, 1)$$
$$\searrow \qquad \swarrow$$
$$\mathrm{Im}(\mathcal{S}) \subset C^\infty(G/K, V_\chi)$$

commutes. Let $(\mathrm{Im}(\mathcal{S}))_K$ and $\mathrm{Im}(\mathcal{B})_K$ denote the space of K-finite vectors in Image $\mathcal{S}$ and Image $\mathcal{B}$ respectively. Since the representations involved are finitely generated admissible modules, then the operator $\mathcal{B}$ extends to a continuous G-equivariant map from $\overline{(\mathrm{Im}(\mathcal{S}))_K}$ to the closure $\overline{(\mathrm{Im}(\mathcal{B}))_K}$.

The analysis in Proposition 7.3 runs into difficulties when ρ_L is dominant singular. As in §6, we first associate to each K-finite function

$$f \in \mathrm{ind}_{M^\# AN}^G(\mathbb{D}(\psi, \lambda) \otimes \xi, \rho_L, 1)$$

a K-finite function

$$F \in \mathrm{ind}_{M_1^\# A_1 N_1}^G(\mathrm{ind}_{P \cap M_1^\#}^{M_1^\#}(\mathbb{D}(\psi, \lambda) \otimes \xi, 0, 1), \rho_L|_{\mathfrak{a}_1}, 1)$$

so that

$$\mathcal{S}f(x) = \mathcal{S}_1 F(x) = \int_K \tau(k)[F(xk)(1)(1)]dk.$$

Next, for each $x \in G, m \in M^\#$, we set

$$\mathcal{S}f(x)(m) = \int_K \tau(k)[f(xk)(m)]dk.$$

$\mathcal{S}f(.)(.)$ is an "extension" of the Szegö map in that $\mathcal{S}f(x)(1) = \mathcal{S}f(x)$. For each $K \cap M^\#$-finite vector Φ in $\mathcal{J}_{M^\#}^\chi$, we consider the inner product

$$\langle \mathcal{S}f(x)(.), \Phi(.) \rangle_{L^2(M^\#)}.$$

Remark 7.4. If Ψ is the function defined by means of

$$\Psi(k)(m_1)(m) = \int_{M^\# \cap K} \tau_\chi^1(m_1 l)^{-1} \tau_\chi(k)^{-1} \Phi(l^{-1}m) dl$$

for $(k, m_1, m) \in (K \times (K \cap M_1^\#) \times M^\#)$, then it is easy to check that Ψ gives rise to a K-finite vector in $\mathrm{ind}_{M_1^\# A_1 N_1}^G(\mathrm{ind}_{P \cap M_1^\#}^{M_1^\#}(\mathbb{D}(\psi, \lambda) \otimes \xi, 0, 1)\rho_L|_{\mathfrak{a}_1}, 1)$. Moreover,

$$\langle \mathcal{S}f(x)(.), \Phi(.) \rangle_{L^2(M^\#)} = \int_K \int_{K \cap M_1^\#} \langle F(xk)(m_1)(.), \Psi(k)(m_1)(.) \rangle_{L^2(M^\#)} dm_1 dk$$
$$= \langle U(P_1, \pi_{\sigma, 0}, \rho_L|_{\mathfrak{a}_1})(x^{-1})F, \Psi \rangle.$$

Proposition 7.5. *We maintain the above assumptions and notation. Let f be a K-finite function in $\operatorname{ind}_{M^\# AN}^G(\mathbb{D}(\psi, \lambda) \otimes \xi, \rho_L, 1)$ and let Φ be a $K \cap M^\#$-finite vector in $\mathcal{J}_{M^\#}^\chi$. Fix $a_{M_1} \in A_{M_1}^+$ and let $a_1 \in A_1^+$. Then,*

$$(8.7) \qquad \lim_{a_1 \to \infty, P_1} a_1^{\rho_1 - \rho_l} \langle \mathcal{S}f(ka_{M_1}a_1)(.), \Phi(.)\rangle_{L^2(M^\#)}$$
$$= \langle A(\bar{P}_1, P_1, \pi_{\sigma, 0}, \rho_L|_{\mathfrak{a}_1})F(ka_{M_1}p), \Psi(p)\rangle.$$

Proof. This is immediate from Remark 7.4 and a result of Langlands. (See for example [K-2, pg.199] or [W-1, 5.3.4].) $\qquad\square$

Proposition 7.6. *For each $\bar{n}_1 \in \bar{N}_1$, let $\Phi_{P \cap M_1^\#}(a_{M_1}, \Phi, F(kp\bar{n}_1)(.))$ be the leading term in the asymptotic expansion of $\langle \pi_{\sigma, 0} F(kp\bar{n}_1)(.), \Phi(.)\rangle$. Then*

$$\lim_{a_{M_1} \to \infty, P \cap M_1^\#} \Big\| \lim_{a_1 \to \infty, P_1} a_{M_1}^{\rho_{M_1}} a_1^{\rho_1 - \rho_L} \langle \mathcal{S}f(ka_{M_1}a_1), \Phi \rangle$$
$$- \int_{\bar{N}_1} \Phi_{P \cap M_1^\#}(a_{M_1}, \tau(p^{-1})\Phi, F(kp\bar{n}_1)(.))d\bar{n}_1 \Big\| = 0.$$

Proof. Similar to Proposition 6.4 $\qquad\square$

Let $\epsilon > 0$, and consider the principal series representation

$$\operatorname{ind}_{M^\# AN}^G(\mathbb{D}(\Psi, \lambda) \otimes \chi, \rho_L + \epsilon\rho_{M_1}, 1).$$

If $f_\epsilon \in \operatorname{ind}_{M^\# AN}^G(\mathbb{D}(\Psi, \lambda) \otimes \chi, \rho_L + \epsilon\rho_{M_1}, 1)$, then define

$$F_\epsilon \in \operatorname{ind}_{M_1^\# A_1 N_1}^G(\operatorname{ind}_{M_1^\# \cap P}^{M_1^\#}(\mathbb{D}(\Psi, \lambda) \otimes \chi, \epsilon\rho_{M_1}, 1), \rho_L|_{\mathfrak{a}_1}, 1)$$

by $F_\epsilon(x)(m) = f_\epsilon(xm)$.

For each $K \cap M^\#$-finite vector Φ in $\mathcal{J}_{M^\#}^\chi$, we have

$$\langle \mathcal{S}_\epsilon f_\epsilon(x)(.), \Phi(.)\rangle_{L^2(M^\#)}$$
$$= \langle U(P_1, \pi_{\mathbb{D}(\Psi, \lambda) \otimes \chi, i\epsilon\rho_{M_1}}, \rho_L|_{\mathfrak{a}_1})(x^{-1})F_\epsilon(.), \Psi(.)\rangle_{L^2(M^\#)}.$$

Here Φ and Ψ are as in Remark 7.4.

In what follows we assume that the f_ϵ transform according to a representation of K (on the left).

Definition 7.7. For a fixed $x \in G$, define the normalized boundary value of $F_\epsilon(x)(.)$ by

$$\mathcal{B}_{M_1} F_\epsilon(x)(m_1)(.)$$
$$= \sum_j \Phi_{P \cap M_1^\#}(\lambda_{\omega_j}, \pi_{\mathbb{D}(.,.) \otimes \chi, i\epsilon\rho_{M_1}}(m_1^{-1})F_\epsilon(x)(1))$$
$$\times \gamma^{-1}(\overline{P} \cap M_1^\#, P \cap M_1^\#, \mathbb{D}(\Psi, \lambda) \otimes \chi, i\epsilon\rho_{M_1^\#})\omega_j(1)(.)$$

with ω_j ranging over an orthonormal basis of the subspace of

$$\operatorname{ind}_{M_1^\# \cap P}^{M_1^\#}(\mathbb{D}(\Psi, \lambda) \otimes \chi, i\epsilon\rho_{M_1}, 1)$$

containing all images in $\operatorname{ind}_{M_1^\# \cap P}^{M_1^\#}(\mathbb{D}(\Psi, \lambda) \otimes \chi, i\epsilon\rho_{M_1}, 1)$ of members of the induced space transforming according to the given K-type.

Then, by Proposition 5.2 we have

$$\mathcal{B}_{M_1}(F_\epsilon(x))(1)(m) = \gamma_{P \cap M_1^{\#}}^{-1}$$
$$\times \sum_j \langle \mathcal{A}(\overline{P} \cap M_1, P \cap M_1, \mathbb{D}(\Psi, \lambda) \otimes \chi, i\epsilon\rho_{M_1}) F_\epsilon(x)(1), \omega_j(1) \rangle_{L^2(M^\#)} \omega_j(1)(m).$$

Definition 7.8. Define the boundary value of $\mathcal{S}_\epsilon f_\epsilon$ by means of

$$\mathcal{B}(\mathcal{S}_\epsilon f_\epsilon)(1) = \mathcal{B}_{M_1}[\sum_j \lim_{(a_1 \to \infty, P)} \langle T(\omega_o) \mathcal{S}_\epsilon(f_\epsilon)(ka_{M_1}pa_1), \omega_j \rangle_{L^2(M^\#)} \omega_j(1)].$$

Thus by Proposition 7.6 and [W-2, page 191] the above is

$$\gamma_{P \cap M_1^\#}^{-1} \times \sum_j \langle \pi_{\mathbb{D} \otimes \chi, i\epsilon\rho_{M_1}}(p) A_P(\omega_o, \pi_{\mathbb{D} \otimes \chi, i\epsilon\rho_{M_1}} \rho_L|_{\mathfrak{a}_1})$$

$$\circ \mathcal{A}(\overline{P} \cap M_1^\#, P \cap M_1^\#, \mathbb{D}(\Psi, \lambda) \otimes \chi, i\epsilon\rho_{M_1}) f_\epsilon(kp)(.), \omega_j(1)(.)\rangle_{L^2(M_1^\#)} \omega_j(1)(.)$$

$$= \sum_j \langle \mathbb{D} \otimes \chi(p) \mathcal{A}_P(\omega_o, \mathbb{D}(\Psi, \lambda) \otimes \chi, \rho_L + i\epsilon\rho_{M_1}) f_\epsilon(k)(.), \omega_j(1)(.)\rangle_{L^2(M^\#)} \omega_j(1)(.).$$

Definition 7.9. If $f \in \text{ind}_{M^\# AN}^G(\mathbb{D}(\Psi, \lambda) \otimes \chi, \rho_L, 1)$ transforms according to a representation of K on the left, then define the normalized boundary value of $\mathcal{S}f$ to be

$$\mathcal{B}[\mathcal{S}f](k) = \lim_{\epsilon \to 0} \mathcal{B}[\mathcal{S}_\epsilon f_\epsilon](k).$$

Thus,

$$\mathcal{B}[\mathcal{S}f](k)(m)$$
$$= \sum_j \langle \mathbb{D}(\Psi, \lambda) \otimes \chi(p) \mathcal{A}_P(\omega_o, (\Psi, \lambda) \otimes \chi, \rho_L) f(k)(.), \omega_j(1)(.)\rangle_{L^2(M^\#)} \omega_j(1)(m).$$

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